International Journal of Research in Applied, Natural and Social Sciences (IJRANSS)
Vol. 1, Issue 2, July 2013, 77-80

TWO IMPORTANT FORMULAE IN THE THEORY OF FINITE GROUPS

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ABSTRACT

In this paper we proposed two formulae to calculate the number of elements of prime order of a finite group. The first one is used to calculate the number of prime order elements of a given direct product representation of an Abelian group for a given positive integer. The second one is used to determine the total number of elements of prime order in a symmetric group S_n of degree *n*.

KEYWORDS: Abelian Group, Direct Product, Prime Order, Symmetric Group

INTRODUCTION

Counting has been one of the greatest challenge to the mankind since ages. To a modern day Mathematician, counting principle is a great challenging interest. Group theory opened the gates of research in counting. Several path breaking formula like Cauchy's, Sylow's theorem etc. are a testimony to it.

In this paper the main objectives are: 1) A method to compute the number of elements of prime order of each prime, which divides the order of the Abelian group, for a given direct product representation of Abelian group for a positive integer and also the total number of prime order elements. 2) Another method to calculate the number of elements of prime order for each prime less than n as well as the total number of elements of prime order in a symmetric group S_n of degree n.

MATHEMATICAL FORMULATION

Problem 1

Let G be a finite group of order $n<\infty$. We know from prime decomposition theorem that any natural number *n* can uniquely factorized (up to order of factorization), i.e., $n = p_1^{m_1} p_2^{m_2} ... p_l^{m_l}, m_1, m_2, ..., m_l \geq 0$, p_i's are primes. We now introduce the notion of partition number function. The partition number function P is a map from N to N (N is the set of all natural numbers), is defined as follows. $P(n)$, where n is any element from N, gives the number of ways in which n can be decomposed in terms of numbers off which at most one of them is equal to n, in descending order, i.e., $n = n_1 + n_2 + ... + n_k, n_1 \geq n_2 \geq ... \geq n_k$.

For example: $P(4)=5$ because the number 4 can be decomposed as $4, 3+1, 2+2, 2+1+1, 1+1+1+1$. We observe that the prime decomposition of n can be ordered into $P(m_1)P(m_2)...P(m_l)$ possible ways. Let the set consisting of all these possible factorization be denoted by Ω. One can see that the above formula also gives the number of Abelian groups of order n upto isomorphism which are the direct product of cyclic groups. Thus $G = \oplus_{\lambda_i} \mathbf{Z}_{\lambda_i}$, where $\mathbf{Z}_{\lambda_i}$ denotes the additive Abelian group of order λ_i, λ_i denotes a non-unit factor of $p_i^{m_i}$. The question is that how many elements of order

p_i (say) exist in the group G with one of the representations from Ω. The number of elements of order p_i, denoted by n_{p_i}, is evaluated here as

$$n_{p_i} = p_i^{N_i} - 1 \tag{1}$$

where N_i the number of group decompositions is involving Abelian groups of order $p_i^r, 1 \le r \le m_i$. Suppose, $p_1, p_2, ..., p_k$ are primes which divides the order of the group. Then the total number of elements of prime order in the given representation of the Abelian group, denoted by n_t, can be determined by

$$n_t = \sum_{i=1}^{k} n_{p_i} = \sum_{i=1}^{k} p_i^{N_i} - k \tag{2}$$

We illustrate the formula by an example. Let G be a group of order $3^4 5^3$. Now for this order the number of Abelian groups upto isomorphism is 15, since $P(4) = 5, P(3) = 3.$ In this case $p_1 = 3, p_2 = 5, m_1 = 4, m_2 = 3.$ Since Ω is the set of all possible factorization of $3^4 5^3$, we choose few such possible factorizations say $3.3.3.3.5^3, 3^2.3^2.5^3, 3^4.5.5.5, 3^2.3^2.5^2 5.$ The Abelian group representation corresponding to these factorizations are $G = \mathbf{Z}_3 \oplus \mathbf{Z}_3 \oplus \mathbf{Z}_3 \oplus \mathbf{Z}_3 \oplus \mathbf{Z}_{5^3}, G = \mathbf{Z}_{3^2} \oplus \mathbf{Z}_{3^2} \oplus \mathbf{Z}_{5^3}, G = \mathbf{Z}_{3^4} \oplus \mathbf{Z}_5 \oplus \mathbf{Z}_5 \oplus \mathbf{Z}_5, G = \mathbf{Z}_{3^2} \oplus \mathbf{Z}_{3^2} \oplus \mathbf{Z}_{5^2} \oplus \mathbf{Z}_5$ respectively. To each case let us now compute the number of elements of order 3 and 5. For the first representation the number N_3 and N_5 corresponding to $p_1 = 3$ and $p_2 = 5$ are 4 and 1 respectively. Hence the corresponding n_3 and n_5 are equal to $3^4 - 1 = 80$ and $5^1 - 1 = 4$ respectively and n_t is $80 + 4 = 84$. Likewise, for the second representation, N_3 and N_5 are 2 and 1 respectively. So, $n_3 = 3^2 - 1 = 8, n_5 = 5^1 - 1 = 4$ and $n_t = 8 + 4 = 12.$ Similarly, for the remaining two cases, $(n_{3,} n_{5,} n_t)$ is equal to $(3^1 - 1, 5^3 - 1, 3^1 + 5^3 - 2), (3^2 - 1, 5^2 - 1, 3^2 + 5^2 - 2)$, i.e., $(2,124,126), (8,24,32)$ respectively.

Problem 2

In this problem the main focus is on the symmetric group S_n of degree n. We derive a formula to calculate the number of elements of prime order in S_n. We know that the order of an element in S_n is the least common multiple of the length of the disjoint cycles of the element. It is obvious that there exist no element of a prime order p in S_n which is greater than n. So the prime order elements are those prime p which are less than n. This is done in the following manner. It is known that if $p < n$ then by division algorithm there exists positive integer m such that $n = mp + r$ with $0 \le r < p$. The number of elements of order p with only one cycle of length p is ${}^{n}P_p / p$. Likewise the number of elements of order p written as product of two disjoint cycles, each of length p, is calculated using the formula $({}^{n}P_p \times {}^{n-p}P_p) / (2! p^2)$. Continuing in this way we find that the number of elements of order p which is represented as m disjoint cycles, each of length p, can be calculated as $({}^{n}P_p \times {}^{n-p}P_p \times ... \times {}^{n-(m-1)p}P_p) / (m! p^m)$. Thus the total number of elements of order p, which is denoted by $n(p)$, is the sum given as follows

$$n(p) = \frac{{}^{n}P_p}{p} + \frac{{}^{n}P_p \times {}^{n-p}P_p}{2! p^2} + ... + \frac{{}^{n}P_p \times {}^{n-p}P_p \times ... \times {}^{n-(m-1)p}P_p}{m! p^m}$$

$$= \sum_{j=1}^{m} \prod_{i=1}^{j} \frac{{}^{n-(i-1)p}P_p}{p \times i} \tag{3}$$

Suppose, $p_1, p_2, ..., p_k$ are primes which are less than n, then the total number of elements of prime order, denoted by T_p, in S_n can be determined by

$$T_p = \sum_{a=1}^{k} n(p_a),$$

$$= \sum_{a=1}^{k} \left[\sum_{j=1}^{m} \prod_{i=1}^{j} \frac{{}^{n-(i-1)p_a}P_{p_a}}{p_a \times i} \right] \tag{4}$$

CONCLUSIONS

In this paper, the derived formula given by Eqs. (1) and (2) are used to compute the number of elements of prime order for each prime as well as the total number prime order elements respectively corresponding to a given direct product representation of a finite Abelian group. Similarly the Eqs. (3) and (4) represents the number of elements of prime order for each prime less than n as well as the total number prime order elements respectively in a symmetric group S_n of degree n. These formulae are useful to calculate the number of prime order elements for the mentioned group.

ACKNOWLEDGEMENTS

The authors acknowledge the suggestions from Dr. Koeli Ghoshal, Department of Mathematics, Indian Institute of Technology Kharagpur, which helped in remarkably improving the presentation of this paper.

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